A New Transformation Invariant in the Orbital Boundary-Value Problem

Richard H. Battin,* Thomas J. Fill,† and Stanley W. Shepperd†
Charles Stark Draper Laboratory, Inc., Cambridge, Mass.

A fascinating corollary to Lambert's famous problem is developed. By applying this new property of two-body orbits, a simple reformulation of the two-point boundary-value problem is possible. This is accomplished by means of a geometrical transformation of the orbital foci which converts the original problem to one for which the initial point is an apsidal point. The elementary form of Kepler's equation then provides the analytic description of the time of flight. The elements of the original orbit are shown to be simply related to the corresponding elements of the transformed orbit. Finally, a simple iterative method of solving the transformed boundary-value problem using successive substitutions is developed. In most cases of interest, convergence is seen to be quite rapid.

Introduction

THERE are a variety[1] of aesthetically appealing geometrical properties associated with the two-body, two-point orbital boundary-value problem. One of these, which is the basis for the main result of this paper, was discovered by Levine[2] over a decade ago in connection with an optical sighting problem for orbital navigation. Levine showed that the true anomaly of the point in an orbit where the velocity vector is parallel to the line of sight from the initial point to the terminal point is independent of the orbit.

Until recently, this result was only of academic interest. Then, it was discovered that the eccentric anomaly of the point where the parallelism occurs is simply the arithmetic mean of the eccentric anomalies corresponding to the end points. This leads to a new corollary of Lambert's theorem in that the radial distance of this point is found to be a function of the same geometrical quantities as the flight time.

The corollary can be exploited by employing a geometrical transformation of the orbital foci which was suggested by Lagrange in a different context. By this means the original boundary value problem is shown to be equivalent to one for which the initial point is an apsidal radius. This radius, as well as the time of flight, are the invariants of the transformation.

The paper concludes with a simple and elegant method for solving the transformed problem. The algorithm is analogous to the well-known method of successive substitutions frequently used to solve the elementary form of Kepler's equation. In this case, however, the technique is universal and applies equally well to elliptic, parabolic, and hyperbolic orbits. In most of the practical cases, the convergence is found to be quite rapid and almost independent of the initial trial solution.

Geometrical Properties

To specify the geometry of the orbital boundary-value problem, denote the initial and target points by P_1 and P_2 and the corresponding position vectors by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ as shown in Fig. 1. Let c be the line of sight distance from P_1 to P_2, and θ, the central angle.

Each orbit connecting P_1 and P_2 with focus at F may be characterized by its corresponding eccentricity vector $\boldsymbol{e}$, sometimes called the Laplace vector, which is directed from F toward pericenter and whose length is the eccentricity e of the orbit. From the equation of the orbit

$$r=p/(1+e\cos f)$$

where p is the orbital parameter and f the true anomaly, we may write

$$\boldsymbol{e}\cdot\boldsymbol{r}_1=p-r_1, \quad \boldsymbol{e}\cdot\boldsymbol{r}_2=p-r_2$$

By subtracting these two equations, we have

$$-\boldsymbol{e}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)/c=(r_2-r_1)/c$$

Thus, the eccentricity vector of any orbit satisfying the boundary conditions has a constant projection on the line P_1P_2. The locus of the termini of such vectors $\boldsymbol{e}$ is a straight line perpendicular to P_1P_2 and at a distance

$$e_s=(r_2-r_1)/c \tag{1}$$

from F as illustrated in Fig. 2.

The orbit whose eccentricity is e_s is, of course, the orbit of smallest eccentricity which satisfies the boundary conditions. It has been called the *symmetric ellipse*[1] since P_1 bears the same relationship to pericenter as P_2 does to apocenter.

Another interesting geometrical property can be demonstrated using the well-known analytic relationship[1] for the eccentricity vector

$$\mu\boldsymbol{e}=\boldsymbol{v}\times\boldsymbol{h}-\mu\boldsymbol{r}/r$$

where μ is the gravitational constant, $\boldsymbol{h}$ the angular momentum vector, and $\boldsymbol{r}$, $\boldsymbol{v}$ the position and velocity vectors of any point on the orbit.

We are concerned with determining that point in the orbit for which the orbital tangent, i.e., the velocity vector, is parallel to P_1P_2. For this purpose we calculate $\boldsymbol{e}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)$ and require $\boldsymbol{v}\times(\boldsymbol{r}_2-\boldsymbol{r}_1)$ to be zero. There results

$$\boldsymbol{r}_0\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)=r_0(r_2-r_1) \tag{2}$$

where $\boldsymbol{r}_0$ is the position vector corresponding to the point where the parallelism occurs. It is surprising to note that this

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*Associate Department Head; also, Lecturer in MIT Department of Aeronautics and Astronautics. Fellow AIAA.

†Staff Member, Guidance and Navigation Analysis Division. Member AIAA.

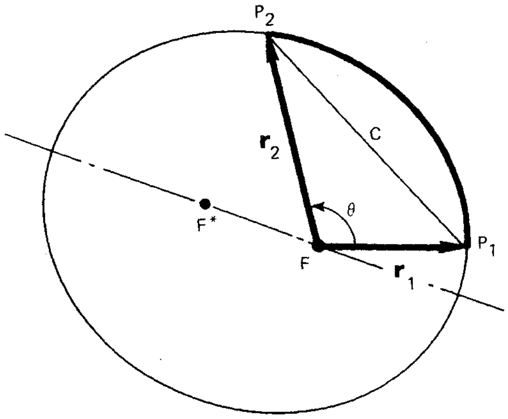

Fig. 1 Geometry of the boundary-value problem.

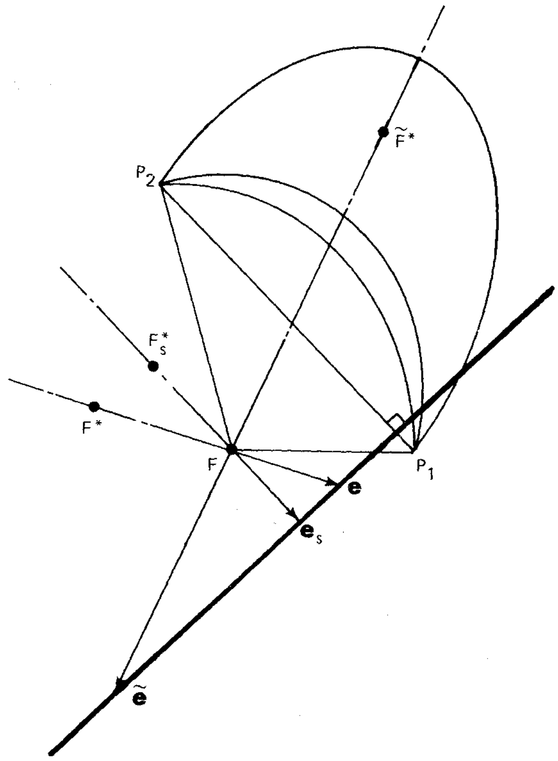

Fig. 2 Locus of the eccentricity vectors.

relation is independent of the orbit and is, therefore, valid for all orbits. If δ is the angle between r_0 and r_1, then we may show that

$$\tan \tfrac{1}{2}\delta = \sin \tfrac{1}{2}\theta / (\sqrt{r_1/r_2} + \cos \tfrac{1}{2}\theta) \tag{3}$$

This property was discovered in an entirely different context by Levine. In his original paper[2] he termed the point in an orbit where the velocity vector is perpendicular to $h \times (r_2 - r_1)$ the *normal point*. Thus, we have the fascinating result that the locus of normal points is a straight line, as shown in Fig. 3.

Corollary to Lambert's Theorem

It is well known[1] that for any orbit, the orbital tangents at P_1 and P_2 and the bisector of the central angle θ have a common point of intersection N. The distance FN, illustrated in Fig. 4, is related to the eccentric anomaly difference of the two termini. Thus, if E_1, E_2 are the eccentric anomalies of a particular orbit, corresponding to the positions P_1 and P_2, we have

$$FN \cos \tfrac{1}{2}(E_2 - E_1) = \sqrt{r_1 r_2} \tag{4a}$$

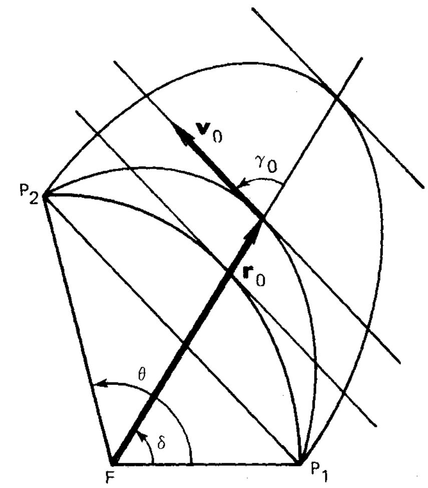

Fig. 3 Locus of the normal points.

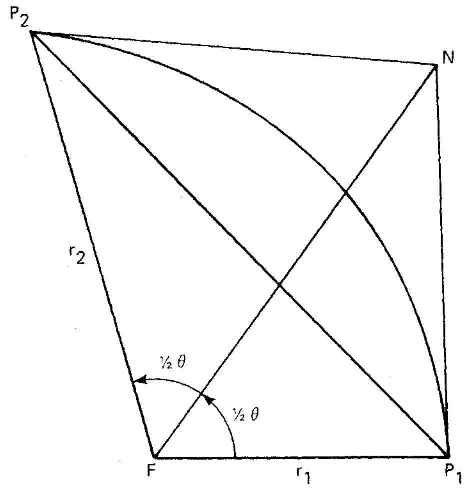

Fig. 4 Orbital tangents and central angle bisector.

For the parabolic orbit the relation is

$$FN = \sqrt{r_1 r_2} \tag{4b}$$

while for hyperbolic orbits

$$FN \cosh \tfrac{1}{2}(H_2 - H_1) = \sqrt{r_1 r_2} \tag{4c}$$

where H is the usual hyperbolic parameter.

In the following we determine the eccentric anomaly of the radius to the normal point. To this end, refer to Fig. 5, where we have constructed the locus of normal points together with two angle bisectors—one bisecting the angle δ and the other bisecting $\theta - \delta$. Also shown in the figure is an arbitrary elliptic orbit, as well as the parabolic orbit connecting P_1 and P_2. Tangents to each of these orbits are constructed at P_1 and P_2 as well as those parallel to P_1P_2. The various points of intersection of these lines are labeled in the figure.

Let r_0 and E_0 denote, respectively, the radius and eccentric anomaly for the normal point on the elliptic orbit. The corresponding radius for the parabola is r_{p0}. Then, by suitably adapting Eqs. (4a) and (4b) to the purpose, we may write for the ellipse

$$FN_1 \cos \tfrac{1}{2}(E_0 - E_1) = \sqrt{r_1 r_0}, \quad FN_2 \cos \tfrac{1}{2}(E_2 - E_0) = \sqrt{r_0 r_2}$$

and for the parabola

$$FN_{p1} = \sqrt{r_1 r_{p0}}, \quad FN_{p2} = \sqrt{r_{p0} r_2}$$

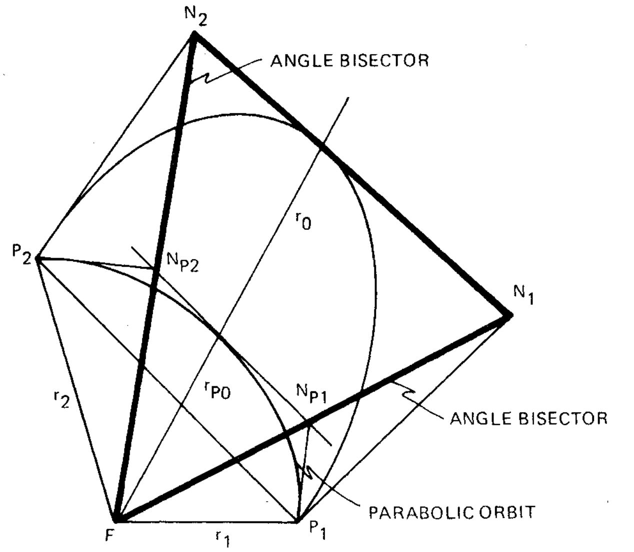

Fig. 5 Geometry for eccentric anomaly of the normal point.

Since triangle $FN_{p1}N_{p2}$ is similar to triangle FN_1N_2, we have

$$FN_{p1}/FN_{p2} = FN_1/FN_2$$

or simply

$$\cos\tfrac{1}{2}(E_0 - E_1) = \cos\tfrac{1}{2}(E_2 - E_0)$$

Hence,

$$E_0 = \tfrac{1}{2}(E_1 + E_2) \tag{5}$$

so that

$$r_0 = a[1 - e\cos\tfrac{1}{2}(E_1 + E_2)] \tag{6}$$

where a and e are the semimajor axis and eccentricity of the ellipse, respectively. Thus, we obtain the fundamental result that the eccentric anomaly of the normal point of an orbit connecting two termini is the arithmetic mean of the eccentric anomalies of these termini. Clearly, an analogous result holds for hyperbolic orbits.

In order to develop further the significance of this important result, we recall Lambert's well-known proposition[1] that the time to traverse the arc from P_1 to P_2 is a function only of the semimajor axis a, the sum of the distances of the initial and final points of the arc from the center of force $r_1 + r_2$, and the chord length c.

A classical result[3] due to Lagrange is that

$$e\cos\tfrac{1}{2}(E_2 + E_1) = \cos\tfrac{1}{2}(\alpha + \beta)$$

where

$$\sin^2\tfrac{1}{2}\alpha = (r_1 + r_2 + c)/4a, \quad \sin^2\tfrac{1}{2}\beta = (r_1 + r_2 - c)/4a$$

Thus, from Eq. (6) we conclude that the radius r_0 depends only on the same three quantities as the flight time, so that

$$r_0 = F(a, r_1 + r_2, c) \tag{7}$$

is established as a new corollary to Lambert's theorem. A similar argument will show that Eq. (7) holds for hyperbolic orbits as well.

Invariance Property

According to Lambert's theorem, if P_1 and P_2 are held fixed, the shape of the orbit may be altered by moving the foci

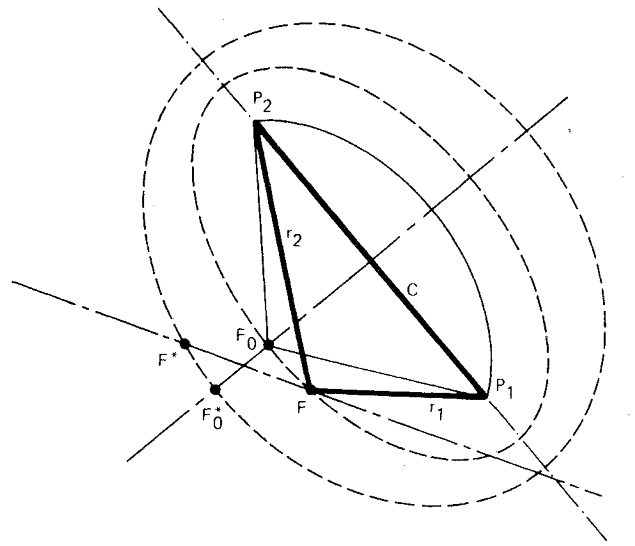

Fig. 6 Geometrical transformation of the boundary-value problem.

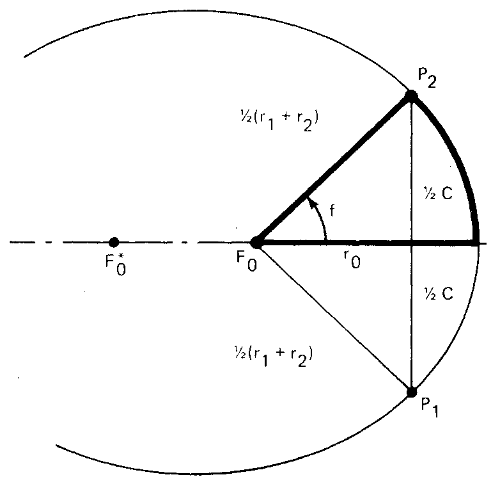

Fig. 7 Transformed version of Lambert's problem.

F and F^* without altering the time of flight, provided, of course, that $r_1 + r_2$ and a are unchanged in the process. The permissible locations for the focus F is an elliptic locus with foci at P_1 and P_2 and with major axis equal to $r_1 + r_2$. Similarly, the locus of the unoccupied or vacant focus F^* is also an ellipse with major axis $4a - (r_1 + r_2)$ and confocal with the elliptic locus of F. Thus, as is shown in Fig. 6, the focus F may be moved to F_0 and the focus F^* to F_0^*; the time to travel the new arc from P_1 to P_2 will be unchanged. But in addition, according to the new corollary, the radius to the normal point in the orbit is also unchanged. Furthermore, the corresponding velocity vector v_0, whose magnitude depends only on r_0 and a and whose direction is parallel to P_1P_2, is unaltered as well by this geometrical transformation.

As shown in the figure and for the application to follow, we purposely choose the transformed orbit‡ so that its major axis is perpendicular to P_1P_2. The geometry of the new orbit is illustrated in Fig. 7.

The time $t - \tau$ to travel from pericenter of the new orbit to the point P_2 is just one half of the time to traverse the original orbit from P_1 to P_2. The pericenter radius is r_0, the terminal radius is $\tfrac{1}{2}(r_1 + r_2)$, and the true anomaly f is related to the original central angle θ according to

$$\cos f = \frac{\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta}{\tfrac{1}{2}(r_1 + r_2)} \tag{8}$$

‡Shepperd[4] discovered this transformed orbit and some of its related properties by analytic manipulation of the equations of the boundary-value problem.

The time of flight equation is the elementary form of Kepler's equation, which, for the ellipse, is simply

$$\sqrt{\mu}(t-\tau)=a^{3/2}(E-e_0\sin E) \tag{9}$$

where§ e_0 and a_0, expressed in terms of the pericenter radius r_0, are given by

$$e_0=\frac{\frac{1}{2}(r_1+r_2)-r_0}{r_0-\sqrt{r_1r_2}\cos\frac{1}{2}\theta}, \quad a=\frac{r_0}{1-e_0} \tag{10}$$

The eccentric anomaly E of the terminal P_2 is obtained from

$$\sin E=\frac{\frac{1}{2}c}{r_0}\sqrt{\frac{1-e_0}{1+e_0}} \tag{11}$$

The preceding discussion must be somewhat modified if the vacant focus F^* of the original orbit is on the opposite side of the chord from F. In that case, the transformation described will result in the radius r_0 being the apocenter distance. This situation may be accounted for analytically in Eqs. (9-11) by the simple artifice of permitting this eccentricity e_0 to be negative. The problem, of course, does not arise for hyperbolic orbits.

The transformed version of Lambert's problem is thus to solve Eqs. (9-11), or the equivalent hyperbolic form, for the pericenter (apocenter) radius r_0. In other words, given a terminal radius F_0P_2. a transfer angle f and a time of flight $t-\tau$, the problem is to determine a conic arc which originates at an apsidal point and satisfies the terminal constraint and flight time. Because of the symmetry, only half of the transformed orbit is involved, so that f and E are confined to the interval between 0 and 180 deg.

Velocity Vector

Solving the transformed version of Lambert's problem will produce directly the velocity vector v_0 which is parallel to the line P_1P_2 of the original orbit. However, for most problems, what is required, of course, is the velocity vector v_1 at the initial point P_1.

For this purpose, we note that v_1 may be written as

$$v_1=\sqrt{\mu}(\sigma_1 r_1+\sqrt{p}\,i_\zeta\times r_1)/r_1^2 \tag{12}$$

where i_ζ is the unit vector normal to the orbital plane and $\sqrt{\mu}\sigma_1=r_1\cdot v_1$. To complete the task, convenient expressions will now be obtained for σ_1 and the orbital parameter p.

The angular momentum h and the quantity σ of the original orbit, each computed at the normal point, are simply

$$h=r_0v_0\sin\gamma_0, \quad \sqrt{\mu}\sigma=r_0v_0\cos\gamma_0$$

where γ_0 is the angle between the position and velocity vectors as shown in Fig. 3. Since γ_0 is the same for all orbits, we may compute it most easily for the symmetric orbit for which the normal point coincides with the extremity of the minor axis. Thus, $\cos\gamma_0=e_s$, where e_s is determined from Eq. (1).

The angular momentum h_0 of the transformed orbit is

$$h_0=r_0v_0$$

§Note that, from Ref. 3, we have also

$$e_0=\cos\phi=e\cos\tfrac{1}{2}(E_2+E_1)=\cos\tfrac{1}{2}(\alpha+\beta)$$

$$E=\psi=\tfrac{1}{2}(E_2-E_1)=\tfrac{1}{2}(\alpha-\beta)$$

Thus, ϕ and ψ are geometrical invariants, as is also the length[3]

$$\sqrt{r_1r_2}\cos\tfrac{1}{2}\theta=a[\cos\tfrac{1}{2}(\alpha-\beta)-\cos\tfrac{1}{2}(\alpha+\beta)]$$

which is the radius to the normal point of the straight line hyperbola[1] connecting P_1 and P_2 $(a=0, e=\infty)$.

because of the invariance of r_0 and v_0. Thus, if p_0 is the parameter of the transformed orbit, we have

$$p=p_0(1-e_s^2), \quad \sigma=\sqrt{p_0}e_s$$

The quantities σ and σ_1 are related through the following well-known identity[1]

$$(r_0\sigma_1+r_1\sigma)\tan\tfrac{1}{2}\delta=\sqrt{p}(r_0-r_1)$$

where δ is determined in Eq. (3). Hence,

$$\sigma_1=\frac{\sqrt{p_0}}{\frac{1}{2}cr_0}\{r_1[r_0-\tfrac{1}{2}(r_1+r_2)]+(r_0-r_1)\sqrt{r_1r_2}\cos\tfrac{1}{2}\theta\} \tag{13}$$

$$\sqrt{p}=\frac{\sqrt{p_0}}{\frac{1}{2}c}(\sqrt{r_1r_2}\sin\tfrac{1}{2}\theta) \tag{14}$$

so that the elements of the velocity vector Eq. (12) are determined.

Solution of the Transformed Problem

A simple iterative method for solving the transformed Lambert problem can be devised using a method of successive substitutions. The iterated variable is chosen to be

$$x^2=\tan^2\tfrac{1}{2}E \tag{15}$$

so that the various quantities appearing in Eq. (9) may be expressed in terms of x and

$$x_0^2=\tan^2\tfrac{1}{2}f \tag{16}$$

as follows

$$e_0=\frac{x_0^2-x^2}{x_0^2+x^2}, \quad \sin E=\frac{2x}{1+x^2}$$

$$\frac{r_0}{\frac{1}{2}(r_1+r_2)}=\frac{1+x^2}{1+x_0^2} \tag{17}$$

For notational convenience, let p_p be the parameter of the parabolic orbit satisfying the boundary conditions and define

$$T=4\sqrt{\mu/p_p^3}(t-\tau) \tag{18}$$

with

$$p_p=\tfrac{1}{2}(r_1+r_2)(1+\cos f)$$

Then Kepler's equation (9) may be written as

$$\frac{\tan^{-1}x}{x}=F(\frac{1}{2},1;\frac{3}{2};-x^2)=\frac{x_0^2-x^2}{w}+\frac{x^2T}{w^{3/2}} \tag{19}$$

where

$$w=(1+x^2)(x_0^2+x^2) \tag{20}$$

and F is a hypergeometric function.

Equation (19) can be written in a more convenient form for our purposes by defining a function G so that

$$F=1/(1+x^2G) \tag{21}$$

Then we have

$$T/w^{1/2}=2+x_0^2+x^2-Gw/(1+x^2G)$$

Next, we replace $x_0^2+x^2$ by $w/(1+x^2)$ and Kepler's equation becomes

$$(1-G)w^{3/2}+2(1+x^2)(1+x^2G)w^{1/2}=(1+x^2)(1+x^2G)T$$

or, alternately, using Eq. (20),

$$(1+G)w^{3/2}+2(1+x^2)(1-x_0^2G)w^{1/2}$$
$$=(1+x^2)(1+x^2G)T$$

In this form, the equation is valid for hyperbolic orbits as well if we identify $-\tanh^2 \frac{1}{2}H$ with x^2. Also, Barker's equation for parabolic orbits results when $x^2=0$ since $G(0)=\frac{1}{3}$.

A universal form of Kepler's equation may thus be expressed as

$$w^{3/2}+3Aw^{1/2}=2B \tag{22}$$

where

$$A=\frac{2(1+\eta)(1-\eta_0 G)}{3(1+G)},\quad B=\frac{(1+\eta)(1+\eta G)T}{2(1+G)} \tag{23}$$

and η, η_0 and w are defined by

$$\eta=\begin{cases}\tan^2 \frac{1}{2}E & \text{elliptic orbits}\\ 0 & \text{parabolic orbit}\\ -\tanh^2 \frac{1}{2}H & \text{hyperbolic orbits}\end{cases}$$

$$\eta_0=\tan^2 \tfrac{1}{2}f$$

$$w=(1+\eta)(\eta_0+\eta)$$

The function $G(\eta)$ is most conveniently evaluated by the method described in the Appendix.

Equation (22) can be solved by a method of successive substitutions. We start with a suitable estimate for η, such as $\eta=\eta_0$ corresponding to a circular orbit or $\eta=0$ for a parabolic orbit, and solve Eq. (22) as a cubic equation in $w^{1/2}$. A new value of η is computed from

$$\eta=\tfrac{1}{2}[\sqrt{(1-\eta_0)^2+4w}-(1+\eta_0)] \tag{24}$$

The process continues until η is no longer changing to within some specified tolerance. The apsidal distance r_0 is then obtained from Eq. (17) as

$$r_0=\tfrac{1}{2}(r_1+r_2)(1+\eta)/(1+\eta_0) \tag{25}$$

Since A and B are never negative, Cardan's solution of Eq. (22) is applicable and we have

$$m=(\sqrt{A^3+B^2}+B)^{1/3},\quad w^{1/2}=m-A/m \tag{26}$$

Alternatively, we can determine $w^{1/2}$ as the cube root of $2B-3Aw^{1/2}$. That is

$$w^{1/2}=[2B-3A\sqrt{(1+\eta)(\eta_0+\eta)}]^{1/3} \tag{27}$$

In almost all instances, convergence is enhanced by solving Eq. (22) for

$$z=\eta_0+\eta \tag{28}$$

Since the equation for z is

$$z^{3/2}+\frac{3A}{1+\eta}z^{1/2}=\frac{2B}{(1+\eta)^{1/2}} \tag{29}$$

then the new η can be obtained from

$$\eta=w/(1+\eta)-\eta_0 \tag{30}$$

with w computed using either Eq. (26) or Eq. (27), whichever is appropriate.

Table 1 Number of iterations for accuracy to eight significant figures

	Elliptic: initial $\eta\sim$ circular orbit								Hyperbolic: initial $\eta\sim$ parabolic orbit					Algorithm
f, deg \ e_0	−0.9	−0.5	0	0.3	0.5	0.7	0.9	1	1	2	5	10	30	
5	5	4	1	3	3	3	3	3	1	3	3	3	3	ZC
25	8	7	1	5	5	5	5	1	1	5	4	4	4	
45	7	8	1	7	7	7	7	7	1	6	6	5	5	
70	5	8	1	9	9	10	10	10	1	9	9	8	6	
80	6	8	1	10	10	11	11	11	1	11	11	10	8	ZC
	25	21	1	18	18	18	17	17	1	14	12	10	7	WC
90	5	7	1	10	11	12	12	12	1	14	21	29	57	ZC
	26	24	1	22	22	21	21	21	1	18	16	13	10	WC
95	5	7	1	11	12	12	13	13	1	18	83	>100		ZC
	28	25	1	23	24	23	23	23	1	21	19	17		WC
	5	10	1	19	22	25	29	29	1	12	[a]	[a]		ZD
	26	22	1	18	18	17	16	16	1	12	[b]	[b]		WD
100	5	9	1	17	20	22	26	27	1	81	[a]			ZD
	27	24	1	21	21	20	20	20	1	17	[b]			WD
110	5	8	1	13	17	19	23	24	1	[a]				ZD
	28	26	1	24	28	28	29	29	1	33				WD
120	5	7	1	11	16	18	22	25						ZD
135	6	5	1	9	12	15	28	(65)[c]						
150	6	6	1	8	10	15	(91)[c]	(102)[c]						
170	6	6	1	6	6	10	(152)[c]	(166)[c]						
175	6	6	1	6	6	7	(>150)[c]	(>150)[c]						

[a] Algorithm diverges regardless of initial η. [b] Algorithm converges only for an initial η near the solution value. [c] *ZD* algorithm fails for initial $\eta=\eta_0$ but converges for starting η near solution value. Numbers in parentheses are for *WD* algorithm for initial $\eta=\eta_0$.

Four possible iterative algorithms¶ emerge and may be summarized as follows:

ZC: Eqs. (26) and (30)
WC: Eqs. (26) and (24)
ZD: Eqs. (27) and (30)
WD: Eqs. (27) and (24)

(The algorithm label *ZC* indicates that Eq. (29) is solved as a cubic equation in $z^{1/2}$ while *ZD* means Eq. (29) is solved for $z^{3/2}$ directly, etc.)

A wide range of possible orbital boundary value problems has been solved by these algorithms, and the results are displayed in Table 1. For the elliptical orbits the initial value of η was chosen to be η_0, corresponding to a circular orbit, while for hyperbolic orbits the initial value was $\eta = 0$, a parabolic orbit. In each case the iteration was continued until η was correct to eight significant figures. (The results would, of course, be improved if the starting value was closer to the actual solution.)

It was found that *ZC* is most efficient, in terms of the number of iterations required, for all orbits with true anomaly f in the range $0 < f < 85$ deg, while *ZD* enjoys this distinction for 110 deg $< f < 180$ deg. The situation is more complicated for 85 deg $< f < 110$ deg. In this intermediate range the *W* algorithms are to be preferred for hyperbolic orbits of moderately high eccentricity. For the remaining orbits, the preferences are not so easily categorized.

A computer program to accommodate all algorithms is easy to devise, and the logic required for an efficient, though not necessarily optimum, selection of the appropriate algorithm need not be complex. Clearly, also, one may switch algorithms during the iteration process as soon as it becomes apparent that the best one was not selected originally.

Appendix

The function $G(\eta)$ may be calculated as a continued fraction[3] using Gauss' theorem for the ratio of two associated hypergeometric functions. A convenient algorithm[3] for this purpose is summarized as

$$\delta_1 = u_1 = \Sigma_1 = 1$$

$$\delta_{n+1} = 1/(1+\gamma_n \eta \delta_n)$$

$$u_{n+1} = u_n(\delta_{n+1} - 1)$$

$$\Sigma_{n+1} = \Sigma_n + u_{n+1}$$

where

$$\gamma_n = \frac{(n+1)^2}{(2n+1)(2n+3)}$$

Successive calculations, for $n = 1, 2, \ldots$, of these equations produces finally $G(\eta)$, since

$$G(\eta) = \tfrac{1}{3} \lim_{n \to \infty} \Sigma_n$$

for $\eta > -1$.

It should be noted that convergence of the continued fraction may be accelerated by applying the readily derived identity

$$G(\eta) = \frac{1}{2(1+\sqrt{1+\eta})}\left[1 + G\left(\frac{\eta}{(1+\sqrt{1+\eta})^2}\right)\right]$$

which may be used recursively to obtain even more rapid convergence with the associated penalty of more complex algebra.

¶Vinh[5] has independently developed a different set of algorithms for this problem. His work motivated the first author in the development of the method presented here.

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